DETERMINANT OF A KNOT AND ITS APPLICATIONS

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ABSTRACT. Building on Dixon's work in "Composite Knot Determinants," this paper explores the knot determinant as a fundamental invariant for classifying mathematical knots. Using linear algebra and topological methods, we rigorously prove the invariance of the knot determinant under labeling conventions, coloring matrix reduction choices, and Reidemeister moves. We then apply this invariant to establish two key results: (1) a knot is p-colorable if and only if an odd prime p divides its determinant, and (2) the determinant of a composite knot equals the product of its components' determinants. These two theorems bridge the combinatorial, algebraic, and topological perspectives in knot classification.

1. INTRODUCTION

Topology studies the properties of geometric spaces that are preserved under continuous deformations [1], [2]. Two objects are topologically equivalent, or homeomorphic, if a continuous function exists with a continuous inverse that maps one object to another bijectively [3], meaning one-to-one and onto, matching every point in one to the other. Intuitively, they can be stretched, twisted, or rotated into each other without tearing, puncturing, or gluing [1], [2]. For example, a circle and a triangle are topologically the same because they can be continuously deformed into each other via stretching and reshaping.

Within topology, a specialized branch called Knot Theory focuses on classifying mathematical knots based on their properties.

Definition 1.1. A mathematical knot is a simple closed curve embedded in R^3.

A knot is the embedding of a circle, S^1, in the 3D Euclidean space, R^3. It is a continuous and closed loop without self-intersections [4], formally expressed as $\gamma: [0,1] \rightarrow R^3$, where $\gamma\ (0) = \gamma(1)$. γ is the image of a continuous embedding in the interval $[0,1]$ where $\gamma\ (0)$ and $\gamma(1)$ are connected ends.

Some segments of knots in R^3 may appear to cross over or under others due to the viewing angle. When projecting onto a 2D plane, we can depict a crossing using an over-strand that passes over another strand that splits into two under-strands. The structure of a crossing is shown in Figure 1.

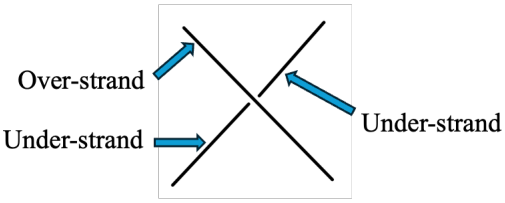


Figure 1. 2D over- and under-strand relationship

One way to label 2D knot diagrams is to label crossings with positive consecutive integers in the set $\{1,2, \dots, n\}$ from the 1st until the nth crossing without reusing any number, then using these numbers to label the strands in the form of $x_{1,}x_{2,} \dots, x_n$. Labeling knots can help us compare and classify knots more easily and mathematically. Figure 2 demonstrates one way to label a trefoil knot, which is a knot that has a minimum of three crossings.

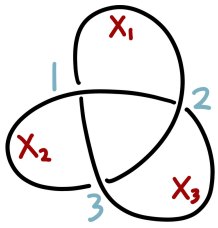


Figure 2. One way to label a trefoil knot

2. KNOT EQUIVALENCE AND INVARIANTS

In Knot Theory, we seek to determine whether two knot projections represent the same knots.

Definition 2.1. Two knots are equivalent, or isotopic, **if and only if** we can deform K_1 into K_2 in R^3 through ambient isotopy, where K_1 is smoothly deformed and never passes through itself. We denote their equivalence as $K_1 \approx K_2$. Ambient isotopy means when two shapes can be transformed into one another without breaking by continuously deforming the entire space they are in [5].

If two diagrams represent isotopic knots, we can transform them into each other by a series of planar isotopies and Reidemeister moves [6].

Definition 2.2. Planar isotopy is the deformation of a knot projection without changing the crossings, as depicted in Figure 3.

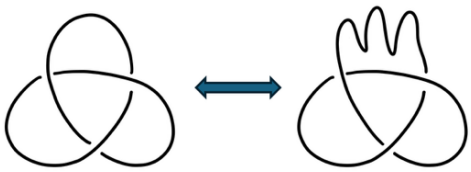


Figure 3. Planar isotopy on a trefoil knot

Definition 2.3. Reidemeister moves are three local knot transformations, as presented in Figure 4. Type 1 is a twist that adds a crossing, while an untwist removes it. Type 2 forms a bigon loop to add two crossings, and an unloop removes them. Type 3 slides a strand completely over or under a crossing, rearranging three crossings.

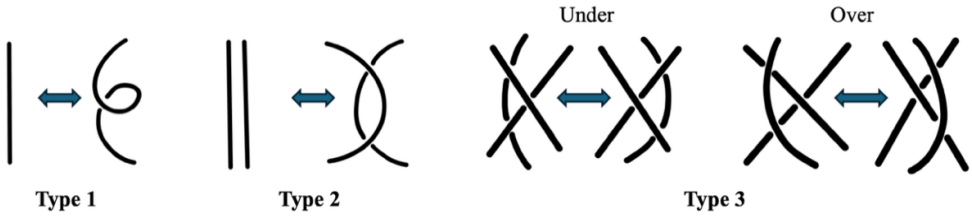


Figure 4. Three Reidemeister moves

To tell whether two knots are topologically different, we must use knot invariants, which could be numbers, polynomials, rings, groups, and more.

Definition 2.4. A knot invariant is a knot property that stays unchanged through any series of planar isotopies and Reidemeister moves.

Two knots are topologically different when they have ≥ **1 different invariants** [4]. We will introduce and prove a knot invariant called the knot determinant.

3. THE KNOT DETERMINANT

To calculate the determinant of a knot, as denoted by $det(K)$, we must construct a coloring matrix from a 2D diagram of this knot.

Definition 3.1. A coloring matrix of a knot with n crossings is an $n \times n$ matrix where each row represents a crossing and each column represents a strand. In each row, 2 represents the over-strand, -1 represents the under-strands, and 0 represents any uninvolved strands.

Special case. In the Type 1 move, for the strand that is both the over-strand and under-strand at a crossing, label 1 for the over-strand section since $2 + (-1) = 1$, and 0 for the under-strand section since -1 is already involved in 1.

Each row in a coloring matrix always sums to 0 since $2 + 2(-1) + 0(n - 3) = 0$ in most cases and $1 + (-1) + 0(n - 2) = 0$ for the case with the Type 1 move. We can multiply certain rows by -1 to enforce all columns sum to 0, while rows still sum to 0 [7]. Each column represents how a single strand interacts with every crossing, which is a row. The sum of the column entries reflects the net influence a strand experiences in the diagram. The zero column sum is a normalization condition we use to create a canonical matrix representation

of the knot. Writing the initial matrix depends on arbitrary choices we make during the diagramming process, including labeling conventions. Hence, multiplying specific rows by -1 effectively adjusts for the arbitrary choices at some crossings. By doing so, we can enforce a zero column sum, thereby removing the bias introduced by our initial choice and ensuring that the final matrix consistently represents the knot. The rows still sum to 0 because we reverse the sign of the original row entries, which reverses the sign of 0 as their sum, and we still obtain 0.

Before calculating the knot determinant, we must know the matrix determinant. It is a scalar value, or a quantity that can be fully described by a single numerical value called a magnitude and that does not contain a direction, calculated from a square matrix. The determinant of a 1×1 matrix equals the only entry it has [8].

To calculate the determinant of a 2×2 matrix, let A be a 2×2 matrix and compute the determinant of A, $det(A)$, using the following method.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

To calculate the determinant of a 3×3 matrix, let B be a 3×3 matrix and compute $det(B)$ as follows. Calculating the determinant of larger square matrices follows the same method. The sample calculation uses the last row $[c_1, c_2, c_3]$ as the reference entries.

$$B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$det(B) = c_1(-1)^{3+1}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + c_2(-1)^{3+2}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3(-1)^{3+3}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$det(B) = c_1(-1)^{3+1}(a_2b_3 - a_3b_2) + c_2(-1)^{3+2}(a_1b_3 - a_3b_1) + c_3(-1)^{3+3}(a_1b_2 - a_2b_1)$$

$$det(B) = c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)$$

Determinants like $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$ are called minors and products like $(-1)^{3+1}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$ are called cofactors. A minor of an entry $a_{i,j}$ in a matrix M is the determinant of the submatrix that remains after R_i and C_j are removed from M [9]. A cofactor is a minor multiplied by -1 raised to the power of the sum of the row and column indices of the reference entry for that minor [10].

If we calculate the determinant of a coloring matrix, it always equals 0 because all the rows are linearly dependent. This means that every row is a multiple of another row or the sum of the multiples of other rows [11]. Since the rows are linearly dependent, the columns are also linearly dependent [11]. Thus, the matrix is singular and its determinant equals 0 [11]. Hence, the determinant of a coloring matrix is a trivial invariant. We will define the knot determinant to obtain a more unique number to represent different types of knots.

Definition 3.2. Let K be a knot and A be its coloring matrix with rows and columns summing to 0. Remove a row and a column from A to obtain M'_K. The knot determinant equals the absolute value of the determinant of M'_K, denoted as $det(K) = |det(M'_K)|$.

Convention. We cannot write a coloring matrix for an unknot with 0 crossings, so we define its determinant as 1. Because if we calculate the determinant of an unknot with ≥ 1 crossings, we obtain $det(unknot) = 1$.

Theorem 3.1. *The determinant of a knot does not depend on the labeling convention imposed on the knot, which row and column to reduce in the coloring matrix, nor which projection we choose for the same knot* [4]*. Therefore, the knot determinant is an invariant.*

Proof. $det(K)$ is independent of the labeling convention.

We need to define four matrices. Let A_j and A_k be coloring matrices obtained by labeling the same knot diagram differently, where crossings and strands are numbered in different orders. Let M'_j be a matrix obtained from A_j by removing a row and a column. Let M'_k be a matrix obtained from A_k by removing the same row and column.

By observation, sometimes M'_j and M'_k are permutations of each other. We can transform them into each other via a finite number of row swaps, column swaps, or both. In linear algebra, a row or column swap switches the position of two rows or columns and flips the sign of the matrix determinant [12]. Hence, we can derive an equation for M'_j and M'_k.

$det(M'_j) = \pm det(M'_k) \cdot (-1)^s$, where s = number of row or column swaps

$det(M'_j)$ and $det(M'_k)$ differ at most by the factor -1 due to the choice of the row and column reduced. Since the knot determinant requires taking the absolute values of $det(M'_j)$ and $det(M'_k)$, we can conclude a crucial equation between them.

$$|det(M'_j)| = |det(M'_k)| \quad (1)$$

Even if M'_j and M'_k are not permutations of each other, Equation 1 remains true because M'_j and M'_k resolve identical arc-crossing relationships. They may have different structures due to which row and column are eliminated, but they are submatrices of the same linear system. Labelings are changed, but not the inherent knot structure. Taking their absolute values, we cancel the sign difference in $det(M'_j)$ and $det(M'_k)$ and achieve equality. Hence, the knot determinant is independent of how we label a knot. ■

Proof. $det(K)$ is independent of which row and column are reduced in the coloring matrix.

We must prove a lemma in linear algebra.

Lemma 3.1. *For any square matrix where **all rows and columns sum to 0**, the absolute value of the determinant of a smaller matrix obtained by deleting one row and one column is invariant under the choice of which row and column are removed* [4].

The rest of this proof mainly follows Dixon's method to prove the lemma [4].

Let A be an $n \times n$ matrix with all rows and columns summing to 0. Let B be an $n \times n$ matrix with 1 for all entries. Let $M'_{i,j}$ be the resulting matrix by removing the i^{th} row, R_i, and the j^{th} column, C_j, from A, where i and j are any row and column in A.

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

$$M'_{i,j} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}$$

We will conduct elementary row and column operations to express $det(A + B)$ in terms of $M'_{i,j}$.

1. Compute $A + B$. Add the corresponding entries from A and B to form $A + B$.

$$A + B = \begin{bmatrix} a_{1,1} + 1 & \cdots & a_{1,j} + 1 & \cdots & a_{1,n} + 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} + 1 & \cdots & a_{i,j} + 1 & \cdots & a_{i,n} + 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} + 1 & \cdots & a_{n,j} + 1 & \cdots & a_{n,n} + 1 \end{bmatrix}$$

2. Add all other rows of $A + B$ to R_i. All entries in R_i become n because all column entries of A sum to 0, and n number of 1s sum to n.

$$A+B=\begin{bmatrix} a_{1,1}+1 & \cdots & a_{1,j}+1 & \cdots & a_{1,n}+1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i-1,1}+1 & \cdots & a_{i-1,j}+1 & \cdots & a_{i-1,n}+1 \\ n & \cdots & n & \cdots & n \\ a_{i+1,1}+1 & \cdots & a_{i+1,j}+1 & \cdots & a_{i+1,n}+1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}+1 & \cdots & a_{n,j}+1 & \cdots & a_{n,n}+1 \end{bmatrix}$$

3. Add all other columns of $A+B$ to C_j. All entries in C_j become n except for $a_{i,j}$, since entries in each row of A sum to 0 and n number of 1s sum to n. For $a_{i,j}$, n number of n's sum to n^2.

$$A+B=\begin{bmatrix} a_{1,1}+1 & \cdots & a_{1,j-1}+1 & n & a_{1,j+1}+1 & \cdots & a_{1,n}+1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1}+1 & \cdots & a_{i-1,j-1}+1 & n & a_{i-1,j+1}+1 & \cdots & a_{i-1,n}+1 \\ n & \cdots & n & n^2 & n & \cdots & n \\ a_{i+1,1}+1 & \cdots & a_{i+1,j-1}+1 & n & a_{i+1,j+1}+1 & \cdots & a_{i+1,n}+1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1}+1 & \cdots & a_{n,j-1}+1 & n & a_{n,j+1}+1 & \cdots & a_{n,n}+1 \end{bmatrix}$$

4. Factor out n from R_i in $det(A+B)$. Entries in R_i become all 1s except for $a_{i,j}$, which becomes n. When we multiply the row of a matrix by x, its determinant is multiplied by x [13]. In Step 3, R_i in $A+B$ has n as a factor in every entry, hence $det(A+B)$ equals n multiplied by the determinant of $A+B$ in Step 3 but with all entries in R_i divided by n.

$$det(A+B)=n\cdot\begin{vmatrix} a_{1,1}+1 & \cdots & a_{1,j-1}+1 & n & a_{1,j+1}+1 & \cdots & a_{1,n}+1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1}+1 & \cdots & a_{i-1,j-1}+1 & \vdots & a_{i-1,j+1}+1 & \cdots & a_{i-1,n}+1 \\ 1 & \cdots & 1 & \vdots & 1 & \cdots & 1 \\ a_{i+1,1}+1 & \cdots & a_{i+1,j-1}+1 & \vdots & a_{i+1,j+1}+1 & \cdots & a_{i+1,n}+1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1}+1 & \cdots & a_{n,j-1}+1 & n & a_{n,j+1}+1 & \cdots & a_{n,n}+1 \end{vmatrix}$$

5. Subtract R_i from all other rows. All entries in the rows other than R_i are subtracted by 1, all entries in C_j except for $a_{i,j}$ become 0 since $n - n = 0$.

$$det(A+B) = n \cdot \begin{vmatrix} a_{1,1} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 1 & \cdots & 1 & n & 1 & \cdots & 1 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

6. Compute $det(A+B)$ in terms of n and $M'_{i,j}$. We can choose C_j as the refence column because its entries are all 0s except for $a_{i,j}$. We only need to multiply one cofactor $(-1)^{i+j} \cdot det(M'_{i,j})$ by the entry n at $a_{i,j}$. The sign of $det(M'_{i,j})$ may differ due to the row and column $a_{i,j}$ is in.

$$det(A+B) = n^2 \cdot (-1)^{i+j} \cdot det(M'_{i,j})$$

We can simplify this expression by taking the absolute value on both sides.

$$|det(A+B)| = |n^2 \cdot det(M'_{i,j})| \qquad (2)$$

In Equation 2, $det(A+B)$ is a integer multiple of n^2 if A is a coloring matrix with all rows and columns summing to 0. $|det(A+B)|$ and n^2 are fixed values for the same coloring matrix, so $|det(M'_{i,j})|$, which represents $det(K)$, is a fixed non-negative integer. Thus, $det(K)$ is invariant no matter which row and column are reduced in the coloring matrix. ■

Proof. $det(K)$ is independent of the choice of the projection of a knot.

We must prove that $det(K)$ is invariant under planar isotopies and Reidemeister moves. We need not prove this for planar isotopies because the coloring matrix stays unchanged after any series of planar isotopies.

We must define four matrices to prove $det(K)$ stays invariant under Type 1 moves. Let A_1be the coloring matrix of the knot's minimal crossing diagram. Let A_2be the post-Type 1 move coloring matrix of the same knot. Make the rows and columns in A_1 and A_2 sum to 0. Let M'_1 be A_1 reduced by 1 row and 1 column. After transforming A_2 into the desired form in Equation 3, obtain M'_2 by reducing 1 row and 1 column in A_2. Ensure the deleted rows and columns do not have 1 on the left diagonal entry and 0s elsewhere.

We can conduct elementary row operations, column operations, or both to transform A_2 into a new form, as depicted in Equation 3. We can swap rows or columns, add a multiple of a row or column to another row or column, or flip the signs of rows or columns while preserving the absolute value of the determinant of a matrix [12].

$$A_2 = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3}$$

Notice that A_1 becomes a submatrix on the top of the left diagonal, 1s occupy the rest of the left diagonal, and 0s are in all other entries. By obversation, $|det(M'_2)|$ equals $|det(M'_1)|$ multiplied by a lot of 1s because the reference entries are on the left diagonal so -1 is always raised to an even power and there is only one minor to multiply, which is $det(M'_1)$. $det(M'_2)$ and $det(M'_1)$ may differ in sign due to which row and column are deleted. We can conclude a relationship between $det(M'_1)$ and $det(M'_2)$.

$$|det(M'_2)| = |det(M'_1)| \tag{4}$$

Equation 4 has the exact format as Equation 3, proving the knot determinant's invariance under Type 1 moves.

We will demonstrate an example to show that the knot determinant is independent of Type 1 moves. Figure 5 displays a labeled trefoil knot before and after a Type 1 move.

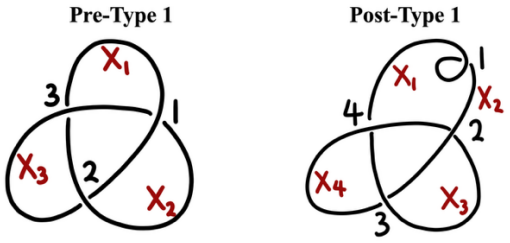


Figure 5. A labeled pre- and post-Type 1 move trefoil knot

Write A_1, M'_1, and A_2 according to the labeling. In the example, M'_1 is A_1 removed by the last row and last column.

$$A_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad M'_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Conduct the elementary row and column operations listed below to transform A_2 into the desired form in Equation 3.

1. Add R_3 to R_4
2. Add C_3 to C_1
3. Add C_2 to C_3
4. Add C_4 to C_2
5. Add R_4 to R_2
6. Add R_4 to R_3
7. Add $-R_1$ to R_3
8. Add $-C_3$ to C_1
9. Add $-R_3$ to R_2

Write the new A_2 and M'_2. In the example, M'_2 is the new A_2 removed by the first row and third column.

$$A_2 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M'_2 = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By computing the $|det(M'_1)|$ and $|det(M'_2)|$, we achieve the following equation.

$$|det(M'_1)| = |det(M'_2)| = 3 = det(trefoil)$$

Thus, the knot determinant is invariant under Type 1 moves.

We will define the same four matrices for Type 2 moves. Note that we cannot always transform A_2 into the structure in Equation 3 because a Type 2 move forces the matrix to combine two new crossings' contributions. It could be impossible to neatly isolate A_1 due to these linked matrix effects. Nevertheless, we can ensure that, outside the isolated submatrix that resembles M'_1 in M'_2, for each row-column pair, at least the row or the column has 1 on the left diagonal and 0s elsewhere. This row or column can be the reference entries used to compute $det(M'_2)$. Once $det(M'_2)$, is found, we will notice that $det(M'_2) = \pm det(M'_1)$. We can obtain Equation 4 by taking the absolute value on both sides. Hence, the knot determinant is independent of Type 2 moves.

A Type 3 move rearranges three crossings, analogous to the three types of elementary row and column operations that preserve the absolute value of the matrix determinant, as mentioned in Type 1 moves. We can try to transform M'_1 and M'_2 into each other. However, M'_1 and M'_2 for Type 3 moves generally cannot transform into each other to match entry-wise. Type 3 moves create small combinatorial differences, altering the combinatorial structure of the matrices in a way that basic row and column operations cannot reconcile. M'_1 and M'_2 at most look similar, but when we compute $det(M'_1)$ and $det(M'_2)$, Equation 4 stays true. Therefore, the knot determinant is independent of Type 3 moves. ■

4. APPLICATIONS OF THE KNOT DETERMINANT

This section will apply the knot determinant to another invariant called p-colorability, and to a knot operation called connected sum.

Definition 4.1. A knot is p-colorable, where p is an odd prime, if we can assign integers from $\{0,1,2,\dots,p-1\}$ to all strands such that at least two distinct labels appear. Each crossing satisfies the relationship $2x - y - z \equiv 0 \ (mod\ p)$, where x represents the over-strand and y and z represent the under-strands. If one knot is p-colorable and the other one is not, these two knots are topologically different.

Theorem 4.1. *For any knot K, K is p-colorable* ***if and only if*** *p divides* $det(K)$*, where p is an* ***odd prime*** [4]*. We denote this as* $det(K) \equiv 0 \ (mod\ p)$*.*

Proof. We need to understand several terms in linear algebra and Knot Theory.

1. The rank of a matrix is the largest integer r where an $r \times r$ submatrix of the matrix has a nonzero determinant [14].
2. The nullity of a matrix A, n, is the dimension of the solution set of the linear system $Ax = 0$ [15]. $Ax = 0$ represents a homogeneous system of linear equations where. We can calculate nullity using the Rank-Nullity Theorem $m = r + n$, where m is the number of columns [15].
3. A constant solution in p-colorability is the trivial solution when we label all strands by c where $c \in \{0,1,2,\dots,p-1\}$.

There are p constant solutions for every knot. These constant solutions span one dimension in the null space because they are scalar multiples of $(1,1,\dots,1)\ mod\ p$. For example, $(4,4,\dots,4)\ mod\ p \equiv 4 \cdot (1,1,\dots,1)\ mod\ p$. The dimension for constant solutions always exists, so we need at least one additional dimension to have non-constant solutions.

This dimension corresponds to nontrivial labels that satisfy $2x - y - z \equiv 0 \ (mod\ p)$ and at least 2 labels are used. Hence, having nontrivial solutions requires the nullity of the coloring matrix to be $\boldsymbol{n \geq 2}$, which contains the trivial solution and at least one non-trivial solution.

Using the Rank-Nullity Theorem, we can calculate that the rank must be $\boldsymbol{r \leq m - 2}$ to have non-constant solutions. Hence, the determinant of the $m \times m$ coloring matrix and the minor of any entry of the coloring matrix, which is the determinant of a $(m-1) \times (m-1)$ matrix, are congruent to 0 $mod\ p$. The determinant of a coloring matrix always equals 0 due to linear dependence from the equation $2x - y - z \equiv 0 \ (mod\ p)$ for every row. The minor of any entry in the coloring matrix and its absolute value are multiples of p. Hence, $p|det(K)$, which ensures $\boldsymbol{n \geq 2}$. The rank drops sufficiently to enable nontrivial solutions. ■

An intriguing remark is that, if $det(K) = p_1 \times p_2 \times \ldots \times p_n$, then K is $p_1, p_2, \ldots, p_n$-colorable since the prime factors $p_1, p_2, \ldots, p_n$ all can divide $det(K)$.

Now we will explore the knot determinant's vital role in connected sum. We first need to comprehend the definition for an oriented knot and a composite knot.

Definition 4.2. An oriented knot is a knot with a specified orientation, corresponding to one of the two ways we can travel along a string, as shown in Figure 6.

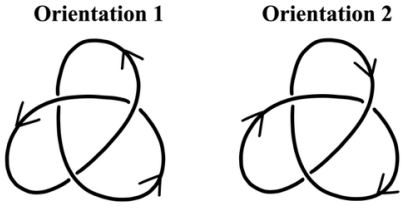


Figure 6. Orientations of a trefoil knot

Definition 4.3. The composite knot $K\#J$ is formed by the sum of two oriented knots K and J that are not isotopic to the unknot, by removing a piece of an outer arc from both knots and

connecting the loose ends without creating new crossings. $K\#unknot \approx K$ since the unknot does not topologically change K to create a new knot.

Theorem 4.2. *For any two knots K and J, the determinant of their composite knot equals the product of the determinant of K and J, denoted as $det(K\#J) = det(K) \times det(J)$* [4].

Proof. We will prove the theorem using an example.

Let K be an oriented left-handed trefoil knot and let J be an oriented right-handed trefoil knot, both having three crossings and are each other's mirror image. Let A_K be an $n \times n$ colorig martix of K and let A_J be an $m \times m$ coloring martix of J. Let $K\#J$ be the oriented composite knot constructed from K and J. This $K\#J$ is called a square knot. Let $A_{K\#J}$ be the $(n+m) \times (n+m)$ coloring matrix of $K\#J$. The labeled diagrams of K, J, and $K\#J$ are included in Figure 7.

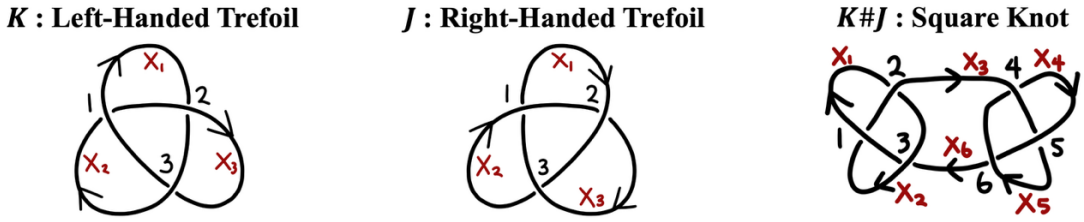


Figure 7. Labeled and oriented diagrams of K, J, and $K\#J$

A_K, A_J, and $A_{K\#J}$ are written as follows. Rows and columns sum to 0 for A_K and A_J.

$$A_K = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \quad A_J = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad A_{K\#J} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \end{bmatrix}$$

The rest of this proof mainly follows Dixon's method [4].

A key observation we can make is that, in $A_{K\#J}$,the top left 3×3 matrix resembles A_K and the bottom right 3×3 matrix resembles A_J. Importantly, $a_{n,n}$ is swapped with $a_{n,n+m}$

and $a_{n+1,n}$ is swapped with $a_{n+1,n+m}$. We can express $A_{K\#J}$ in the form in Equation 5, where entries outside A_K and A_J are mostly 0s.

$$A_{K\#J} = \begin{bmatrix} A_K & 0 \\ 0 & A_J \end{bmatrix} \tag{5}$$

To find $det(K\#J)$, we first need to delete a row and a column in $A_{K\#J}$. The row will be R_{n+1} and the column will be C_{n+m} to help prove the theorem. After deletion, we have a $(n + m - 1) \times (n + m - 1)$ matrix that resembles the one in Equation 6.

$$M'_{K\#J} = \begin{bmatrix} A'_K & 0 \\ 0 & M'_J \end{bmatrix} \tag{6}$$

$M'_{K\#J}$ is $A_{K\#J}$ reduced by 1 row and 1 column, A'_K is a matrix that looks similar to A_K, and M'_J is A_J reduced by R_{n+1} and C_{n+m}.

A'_K appears as an $n \times n$ matrix with 0 at $a_{n,n}$. The original under-strand there is cut and becomes a part of the bridging arc between the two trefoils, we label 0 instead of -1. During the connected sum, the under-strand is cut and reconnected, so it no longer serves as an under-strand at the merged crossing in the composite knot's matrix. Its entry becomes 0.

$$A'_K = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

We need to prove $|det(A'_K)| = det(K)$. We will mainly follow Dixon's method [4].

Proof. For any knot K, $|det(A'_K)| = det(K)$.

We need to use cofactor expansion calculate $det(A'_K)$, which is a way to calculate the determinant of a matrix. We can expand A'_K along R_n to match Equation 7.

$$det(A'_K) = \sum_{j=1}^{n} [a_{n,j} \cdot (-1)^{n+j} \cdot det(A''_{K,(n,j)})] \tag{7}$$

$A''_{K,(n,j)}$ is A'_K removed by R_n and C_j, $det(A''_{K,(n,j)})$ is the minor for the reference entry $a_{n,j}$, and $(-1)^{n+j} \cdot det(A''_{K,(n,j)})$ is the cofactor for $a_{n,j}$. We can take absolute values on both sides of Equation 7 to cancel the effect of $(-1)^{n+j}$. Any $A''_{K,(n,j)}$ will not contain the 0 at $a_{n,n}$ of A'_K, hence $det(A''_{K,(n,j)})$ is equivalent to $det(M'_K)$ up to sign where M'_K is A_K removed by any row and column. Since $|det(M'_K)| = det(K)$, we can substitute $|det(M'_K)|$ by $det(K)$ to obtain Equation 8.

$$|det(A'_K)| = |\sum_{j=1}^{n} [a_{n,j} \cdot det(K)]| \tag{8}$$

Using the square knot and Equation 8 to compute $|det(A'_K)|$, we obtain Equation 9.

$$|det(A'_K)| = |(-1) \cdot det(K) + 2 \cdot det(K) + 0 \cdot det(K)|$$

$$|det(A'_K)| = det(K) \tag{9}$$ ■

Using Equation 6, we can express $det(M'_{K\#J})$ as follows in Equation 10. $M'_{K\#J}$ is a block-triangular matrix, so its determinant equals the product of the determinants of its diagonal blocks [4], which are A'_K and M'_J.

$$|det(M'_{K\#J})| = |det(A'_K) \cdot det(M'_J)| \tag{10}$$

But substituting $det(K\#J)$ for $det(M'_{K\#J})$, $det(K)$ for $det(A'_K)$, and $det(J)$ for $det(M'_J)$, we can rewrite Equation 10 as follows and finish the proof.

$$det(K\#J) = det(K) \cdot det(J)$$ ■

5. CONCLUSION

This paper introduced the basics of knot theory and fixated on the knot determinant, a key invariant to differentiate mathematical knots. We proved the invariance of the knot

determinant under the choice of labeling, row and column reduction in the coloring matrix, and the choice of projection. We demonstrated its utility by applying it to p-colorability and knot composition to prove two vital theorems in Knot Theory. First, a knot's p-colorability must divide its knot determinant where p is an odd prime. Second, a composite knot's determinant can be calculated by multiplying its components' determinants. These findings connect linear algebra and topological approaches, offering a framework for future work on knot classifications, such as higher-dimensional invariants, and computational knot theory.

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