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Surfaces and Invariants: To what extent does the output of Seifert's algorithm depend on diagram complexity, and under what structural conditions does the canonical genus diverge from the true knot genus?

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1. INTRODUCTION

Tangled structures have captivated humans for millennia, with knots applied in maritime navigation, textile production, and decorative art. The mathematical study of knots originated in the 19th century when Lord Kelvin hypothesized that chemical elements were composed of knotted vortices in the luminiferous aether (Thomson). Although scientists later abandoned this hypothesis, the mathematical framework for classifying knots developed into a branch of low-dimensional topology called knot theory. Knot theory analyzes the intrinsic properties of 1D closed curves embedded in the 3D Euclidean space, with the main goal of telling whether two knots are topologically different.

My personal engagement with knot theory began during a research summer program in Grade 10, where I was first exposed to Seifert's algorithm. This geometric procedure takes a 2D planar projection of a knot and builds a bounded, orientable 3D surface. When I applied this algorithm to a standard three-crossing diagram of a trefoil knot, it created a surface with one handle, corresponding to a genus of one. However, when I redrew the same knot with a strand sliding over another, the algorithm created a surface with two handles, indicating a genus of two. This seemed contradictory because the topological complexity, which stays the same under continuous deformations, of a knot surface is meant to be a fixed knot invariant, but the output varied just because of how the diagram was drawn.

The tension between the intrinsic complexity of a mathematical object and the arbitrary representations used to study it inspires this paper. This essay investigates the research question: **To what extent does the output of Seifert's algorithm depend on diagram complexity, and under what structural conditions does the canonical genus diverge from the true knot genus?**

To address this question, I use a comparative case-study methodology across three different knots. First, I analyze the trefoil knot 3_1 to quantify how diagrammatic complexity due to redundant crossings directly increases the algorithm's output. Second, I examine the figure-eight knot 4_1 to determine whether geometric intricacy, specifically hyperbolicity, is sufficient to undermine the algorithm's effectiveness on minimal diagrams. Third, I investigate the Whitehead double of the trefoil $Wh(3_1)$ to evaluate how satellite knot constructions force a divergence

between the canonical genus, which is the minimum genus obtained using Seifert's algorithm, and the true genus.

This investigation matters mathematically for three reasons. First, it tests the boundaries of Seifert's 1934 surface construction (Seifert), testing when this topological tool succeeds or fails. Second, the divergence between the canonical genus and the true genus indicates a deeper issue in mathematics, that the risk of mistaking an incidental product of a chosen projection for a true topological invariant. Third, by evaluating a counterexample, the Whitehead double, this connects basic visual topology to advanced special embeddings. This explains why mathematicians must use both visual algorithms and modern algebraic invariants to classify knots.

2. MATHEMATICAL FOUNDATION

A mathematical knot K is defined as a continuous, non-self-intersecting embedding of the circle S^1 into the 3D Euclidean space $\mathbb{R}^3$. Because K is an 1D loop, visualizing it requires projecting its shadow onto a 2D plane. This produces a knot diagram, which contains planar curves punctuated by crossings, each with an overpass and an underpass. Two knots are topologically equivalent if one can be continuously deformed into the other without cutting the curve or passing through itself. This deformation is ambient isotopy (Adams).

When working with diagrams, ambient isotopies in 3D space are captured by three local diagrammatic movements called Reidemeister moves (Adams).

Type 1 is a twist that creates a crossing, while untwisting removes it (Fig. 1).

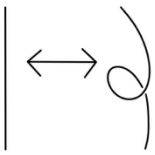


Figure 1: Type 1 Reidemeister Move

Type 2 forms a bigon, a two-sided region, by sliding one strand over another, creating two crossings (Fig. 2). An unloop removes them.

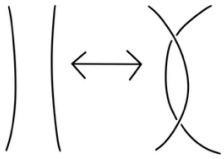


Figure 2: Type 2 Reidemeister Move

Type 3 slides a strand fully over or under a crossing, rearranging three crossings (Fig. 3).

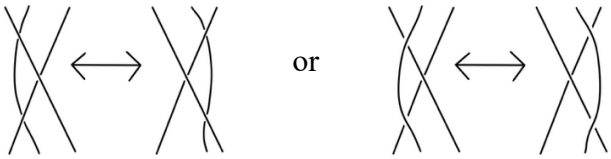


Figure 3: Type 3 Reidemeister Move

Reidemeister's theorem guarantees that two diagrams represent ambient isotopic, or identical, knots if and only if their diagrams can be transformed into each other through a finite sequence of these moves.

We are interested in alternating and non-alternating knots when investigating whether non-alternating knots cause the canonical genus to be greater than the true genus. A knot diagram is alternating if, as one travels along the strand, the strand goes over a crossing, then under the next, over the next, and so on. The crossings must alternate between over and under throughout the diagram (Weisstein). Many knots, including the trefoil and figure-eight, have alternating diagrams. Alternating structure has significant influence on the Seifert's algorithm, as shown in the case study.

A foundational geometric tool for analyzing a knot's topological structure is the Seifert surface. For an oriented knot K, which means a knot marked with one direction along its strands, a Seifert surface S is a compact, connected, and oriented surface embedded in $\mathbb{R}^3$ such that its boundary ∂S is exactly K. S being oriented means that it contains two well-defined, distinct sides, where one is the top, and the other is the bottom. Non-orientable surfaces, such as the Möbius strip, are not Seifert surfaces.

The topological complexity of a compact, orientable surface with one boundary component, which means the surface has exactly one edge, which is the knot itself, is quantified by its genus g, which intuitively means the number of handles attached to the surface. Because any given knot bounds an infinite number of distinct Seifert surfaces, topologists define the knot genus, or true

genus, $g(K)$, as the absolute minimum genus among all Seifert surfaces spanning K (Brittenham and Jensen), whether generated by Seifert's algorithm or not.

$$g(K) = \min\{g(S) | S \text{ is a Seifert surface for } K\}$$

The true genus is strictly a topological invariant because it is fixed for a given knot, regardless of the 2D projection.

A related invariant is the free genus, $g_f(K)$, which restricts the genus to Seifert surfaces whose complements in 3D space are topologically simple. Denoting the 3D sphere as S^3, the complement of a Seifert surface S is the space $S^3 \backslash S$ left behind after removing the surface. A Seifert surface is defined as free if its complement $S^3 \backslash S$ is a handlebody. Equivalently, in algebraic topology terms, the fundamental group of the complement is a free group (Nakamura). A handlebody is topologically simple, it is like a solid sphere with non-intersecting solid handles attached to its surface (Fig. 4). If the empty space left behind by removing the Seifert surface can be continuously deformed into such a solid, the surface is free. Thus, $g_f(K)$ measures the minimum genus among all free Seifert surfaces spanning K.



Figure 4: A Three-Handled Handlebody

Finally, the canonical genus $g_c(K)$ is defined as the minimum genus among all Seifert surfaces generated by Seifert's algorithm over every possible diagram D representing the knot K.

$$g_c(K) = \min\{g_c(D) | D \text{ is a diagram of } K\}$$

For any knot K, these three quantities satisfy the inequality $g(K) \leq g_f(K) \leq g_c(K)$ (Nakamura). The first inequality $g(K) \leq g_f(K)$ holds because $g(K)$ minimizes over the set of all Seifert surfaces, whereas $g_f(K)$ minimizes over a restricted subset with only free surfaces. The minimum of an unrestricted set must be less than or equal to the minimum of a restricted subset. The second inequality $g_f(K) \leq g_c(K)$ is true because every Seifert surface constructed via

Seifert's algorithm is proven to have a handlebody complement, making every Seifert algorithmic surface a free surface (Ozawa). Therefore, $g_c(K)$ minimizes over a specific set of surfaces within the set of all free surfaces. This essay investigates the conditions under which these genera become unequal.

3. METHODOLOGY

A computational methodology is chosen to investigate how diagrammatic complexity affects algorithmic output and under what conditions $g_c(K)$ exceeds $g(K)$. We apply Seifert's algorithm on chosen projections, calculate the genus of the constructed surfaces using a derived formula, and evaluate the outputs against established invariants from peer-reviewed literature and the KnotInfo database.

Seifert's algorithm constructs an orientable surface S_D from any oriented knot diagram D through three steps (Adams). We use the oriented trefoil knot as an example.

1. **Resolution.** We eliminate each crossing by smoothing it according to the orientation (Fig. 5). Following the direction of the arrows, the over-strand is connected to the under-strand so that both resulting arcs continue smoothly without intersecting. The crossing is cut and reconnected so that the two strands pass alongside each other instead of crossing. This process removes all d crossings and decomposes the diagram into a set of s disjoint, non-intersecting curves called Seifert circles.

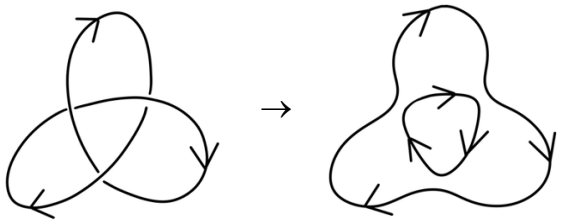


Figure 5: Resolving Three Crossings into Two Seifert Circles

2. **From circles to disks.** Each of the s Seifert circles is assigned a distinct height elevation in $\mathbb{R}^3$ and capped to form a flat, non-intersecting 2D disk (Fig. 6).

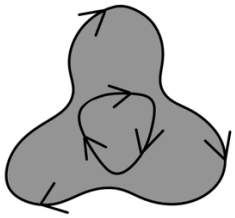


Figure 6: Filled Circles Become Two Disks at Different Heights

3. **Bands.** At the location of each original crossing, a half-twisted rectangular band is attached to connect the elevated disks, reconstructing the knot as the boundary ∂S_D of the resulting surface (Fig. 7).

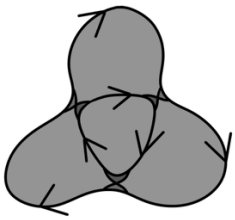


Figure 7: Three Half-Twisted Bands Connect the Disks

To calculate the genus $g_c(D)$ of the surface produced from a diagram D with d crossings and s Seifert circles, we use the Euler characteristic, a topological invariant that captures the shape of a surface. The Euler characteristic χ is defined through triangulation, meaning we divide a surface into polygons and count $\chi = V - E + F$, where V, E, and F are the numbers of vertices, edges, and faces. χ is independent of how the surface is triangulated.

Instead of arduously triangulating the entire Seifert surface, we track how χ changes as we build the surface through Seifert's algorithm.

1. **Circles become disks.** After resolving the d crossings, we obtain s disjoint Seifert circles, which we fill with disks. Each disk has Euler characteristic 1 because a disk can be triangulated as a triangle with $V = 3$, $E = 3$, and $F = 1$, giving $\chi = 1$. Since we have s disks, the total Euler characteristic at this stage is $\chi_{\text{initial}} = s$.
2. **Attach bands.** We attach half-twisted rectangular bands to the location of the original crossings. When a band is connected to the disks, two of its edges and all four of its vertices are attached to the disk boundaries. The additions are the band's interior face

and two unattached edges. Thus, each band changes χ by $\Delta\chi = \Delta V - \Delta E + \Delta F$, which is $\Delta\chi = 0 - 2 + 1 = -1$. Attaching d bands reduce the Euler characteristic by d, giving $\chi(S_D) = s - d$.

3. **Relate Euler characteristic to genus.** For any connected, orientable surface with one boundary component, the Euler characteristic relates to genus g by $\chi = 1 - 2g$ (Davidian and Gong). A sphere is a closed surface with no handles, so it has $g = 0$. Its Euler characteristic can be computed via triangulation. For instance, a tetrahedron representing the sphere has $V = 4$, $E = 6$, and $F = 4$, giving $\chi = 4 - 6 + 4 = 2$. Adding a handle, which means increasing g by 1, requires removing two disks, reducing χ by 2. Thus a closed surface with g handles has $\chi = 2 - 2g$. The Seifert surface is bounded by the knot, so creating one boundary component requires removing one open disk. Removing an open disk removes a face, and since a disk has $\chi = 1$, $\chi(S_D)$ decreases by 1. Hence, $\chi(S_D) = \big(2 - 2g_c(D)\big) - 1 = 1 - 2g_c(D)$.
4. **Derive formula.** Equating the $\chi(S_D)$ expressions, we have $s - d = 1 - 2g_c(D)$. Solving for $g_c(D)$, we obtain the genus formula $g_c(D) = \frac{d-s+1}{2}$.

This is the central formula of the essay. It indicates that the genus of a surface constructed by Seifert's algorithm depends on the number of crossings, d, and the number of Seifert circles, s. In the case studies, this formula is used to compute $g_c(D)$ for minimal and non-minimal diagrams, comparing the outputs to established values of $g(K)$ and $g_f(K)$.

A comparative case study can isolate the three variables identified in the research question. The trefoil knot 3_1 tests the effect of crossing number by comparing a minimal diagram, where $d = 3$, against a non-minimal diagram with added crossings, where $d = 5$, quantifying how diagram complexity increases $g_c(D)$. The figure-eight knot 4_1 tests whether hyperbolic geometry can undermine the algorithm on minimal diagrams, being the control variable that isolates the role of alternating structure on the genus calculation. The Whitehead double $Wh(3_1)$ tests whether satellite construction and non-alternating structure cause $g_c(K) > g(K)$, even with a minimal diagram. This essay compares the computed $g_c(D)$ against established genus values on KnotInfo, and Tripp and Brittenham for satellite and non-alternating genus theorems. These sources provide

verified knowledge necessary to determine when Seifert's algorithm succeeds or fails relative to the true genus.

4. CASE STUDY 1: TREFOIL KNOT (3_1) FOR DIAGRAM DEPENDENCE

The trefoil knot 3_1 case quantifies how diagrammatic complexity influences algorithmic output. The trefoil is the simplest knot that cannot be untangled into a loop with no crossings. It has a minimal crossing number of $c(3_1) = 3$. Its true genus is $g(3_1) = 1$ (Livingston and Moore) on KnotInfo.

We first apply Seifert's algorithm to the standard minimal alternating diagram $D_{\min}$ of the trefoil knot (Fig. 5). This diagram contains $d = 3$ crossings. Resolving all crossings according to the knot's orientation produces $s = 2$ Seifert circles (Fig. 6). Applying the derived formula gives:

$$g_c(D_{\min}) = \frac{3 - 2 + 1}{2} = 1$$

On the minimal diagram, Seifert's algorithm constructs a surface of genus 1, matching the true genus $g(3_1) = 1$. From the inequality $g(K) \leq g_f(K) \leq g_c(K)$, we can infer that its free genus $g_f(3_1)$ also equals 1.

Now, we introduce artificial diagrammatic complexity. Consider a non-minimal diagram D_{nm} created by applying a Type 2 Reidemeister move to two parallel strands of the trefoil with the same orientation (Fig. 8).



Figure 8: Trefoil with Type 2 Reidemeister Move

This local alteration slides one strand over the other, creating two adjacent crossings without altering knot's topological structure. The crossing number increases to $d = 5$. When the move is applied, it creates a pair of crossings where one strand passes over the other, then under it. Resolving these two crossings produces arcs that connect the existing Seifert circles without

forming new closed loops. These arcs extend the boundaries of the existing Seifert circles rather than closing new circles. Thus, the Seifert circle count remains $s = 2$. Applying the formula:

$$g_c(D_{\text{nm}}) = \frac{5 - 2 + 1}{2} = 2$$

This result indicates diagram dependence. Although D_{min} and D_{nm} represent ambient isotopic representations of the trefoil, Seifert's algorithm produces a genus-1 surface from D_{min} and a genus-2 surface from D_{nm}. The canonical genus of the knot itself $g_c(3_1)$ remains 1 because it takes the minimum output across all diagrams, but the output $g_c(D)$ for an input diagram is increased by redundant crossings.

Each new crossing added by the Type 2 move causes the algorithm to attach an additional half-twisted band. Because the bands do not generate new Seifert circles to balance the calculation, d increases while s remains constant, increasing $g_c(D)$. The proves that Seifert's algorithm is sensitive to planar representation.

5. CASE STUDY 2: FIGURE-EIGHT KNOT (4_1) FOR ALTERNATING CONTROL

Having established that non-minimal diagrams increase algorithmic output, we use the figure-eight knot 4_1 to investigate whether geometric intricacy alone causes Seifert's algorithm to fail on a minimal diagram.

The figure eight knot is a prime, alternating knot with a minimal crossing number of $c(4_1) = 4$. Its true genus is $g(4_1) = 1$ (Livingston and Moore). However, unlike the trefoil, it is a hyperbolic knot (Krause and Sharma).

To understand hyperbolicity, we examine the knot complement $S^3 \backslash 4_1$, which is the three-dimensional space surrounding the figure-eight. A knot is hyperbolic if its complement admits a complete Riemannian metric of constant negative curvature (Thurston). In differential geometry, constant negative curvature means the surrounding space locally behaves like a multi-dimensional saddle point at every location. Geodesic lines, the shortest paths through space, diverge from one another exponentially, and the interior angles of geometric triangles always sum to <180°.

Mostow's Rigidity Theorem states that the geometric properties of a hyperbolic knot's complement, such as its volume, are completely determined by the knot's topological structure

(Zhang). This means hyperbolic knots cannot be split into simpler pieces along embedded surfaces. The complex geometry is the knot's inherent feature regardless of how its drawn. We analyze the figure-eight to test whether this complexity interferes with Seifert's algorithm's ability to find the minimal surface.

We apply the algorithm to the standard minimal alternating diagram $D_{\min}$ of the oriented figure-eight knot (Fig. 9). This diagram has $d = 4$ crossings. Resolving these crossings along their orientation arrows produces $s = 3$ Seifert circles.

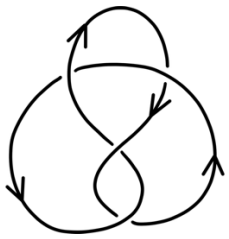


Figure 9: Minimal Alternating Diagram of Figure-Eight

Applying the formula gives $g_c(D_{\min}) = \frac{4-3+1}{2} = 1$.

Despite the complex hyperbolic geometry of the figure eight's complement space, Seifert's algorithm produces $g_c(D_{\min}) = 1$, equaling the true genus $g(4_1) = 1$.

This case study shows that geometric intricacy alone, in this case hyperbolicity, is insufficient to diverge the canonical genus from the true genus on minimal diagrams. The structural feature preserving equality is the alternating nature of the diagram. In an alternating diagram, over-crossings and under-crossings alternate consecutively along the full strand length, which guarantees that Seifert's algorithm always identifies the minimal genus surface (Murasugi). To find the gap where $g_c(K) > g(K)$ across all diagrams, we must examine non-alternating constructions.

6. CASE STUDY 3: WHITEHEAD DOUBLE OF TREFOIL ($Wh(3_1)$) FOR THE GAP

To observe a divergence where the canonical genus strictly exceeds the true genus $g_c(K) > g(K)$ across every possible diagram, we take the Whitehead double of the trefoil, $Wh(3_1)$.

To define the Whitehead double, we adopt the formal definition of satellite knots in The Knot Book by Colin Adams (Adams).

1. Let K_1 be a knot inside an unknotted solid torus V (Fig. 10). A solid torus V is a 3D space topologically equivalent to $S^1 \times D^2$, which means a filled ring.

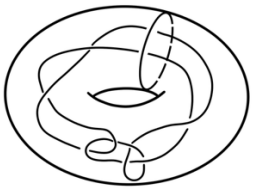


Figure 10: Knot K_1 inside solid torus V

2. We take the solid torus V, which contains K_1, and continuously deform V into the shape of a nontrivial second knot K_2, the companion knot. In Figure 11, K_2 is a trefoil.

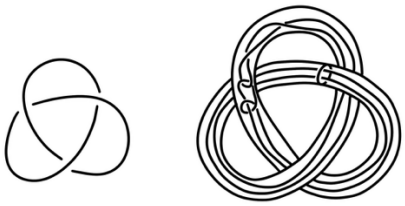


Figure 11: V deformed into the shape of K_2

3. This deformation maps the internal knot K_1 into a new, complex knot K_3 inside $\mathbb{R}^3$. K_3 is called the satellite knot of the companion K_2. K_1 intersects every meridional disk of V and cannot be isotope to lie inside a small, unknotted ball inside V.

A Whitehead double is a specific type of satellite knot where the pattern knot K_1 inside the unknotted solid torus V is an unknot that has been twisted so it clasps itself once, which somewhat resembles the Whitehead link (Fig. 12).

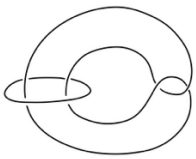


Figure 12: Whitehead link

When this torus is tied into the trefoil shape, which acts as the companion K_2, the resulting knot is the Whitehead double of the trefoil $Wh(3_1)$ (Fig. 13).

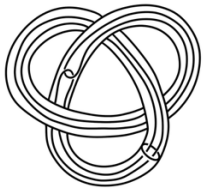


Figure 13: Whitehead double of trefoil $Wh(3_1)$

The Whitehead double $Wh(3_1)$ is non-alternating and satellite. These properties each cause $g_c(K) > g(K)$ though they may appear together. Non-alternating diagrams cause Seifert's algorithm to produce extra Seifert circles and bands, while the satellite knot hiding a simpler surface inside the companion torus makes the surface invisible to the algorithm. The Whitehead double contains both properties that create the gap between $g_c(K)$ and $g(K)$.

The true genus of the Whitehead double of the trefoil is $g\big(Wh(3_1)\big) = 1$ (Tripp) because the knot bounds a genus-1 ribbon surface, and its free genus is $g_f\big(Wh(3_1)\big) = 2$ (Gruber). However, the canonical genus is strictly larger. Tripp proved a general theorem establishing that for the Whitehead double of any knot K, the canonical genus satisfies:

$$g_c\big(Wh(K)\big) = 2g(K) + 1$$

Because the companion trefoil has true genus $g(3_1) = 1$, substituting this gives:

$$g_c\big(Wh(3_1)\big) = 2(1) + 1 = 3$$

The genera all differ, representing a significant failure of the algorithm. It proves that even if we minimize Seifert's algorithm across every single one of infinitely many valid diagrams of $Wh(3_1)$, it cannot produce a surface with a genus less than 3.

Seifert's algorithm operates locally on 2D projections. When processing the Whithead double diagram, the algorithm treats the localized crossings of the Whitehead clasp as essential topological features. Resolving clasp crossings creates additional Seifert circles and half-twisted bands, increasing the canonical genus. The algorithm is blind to the global spatial geometry,

meaning it cannot detect that the true genus-1 trefoil surface is embedded inside the surrounding companion torus. In sum, non-alternating satellite structures cause $g_c(K) > g(K)$.

7. EVALUATION

This investigation shows both the strengths, mainly being simple and useful, and the weaknesses, mainly structural limitations, of Seifert's algorithm as a tool for computing knot genus.

The formula $g_c(D) = \frac{d-s+1}{2}$ is straightforward and simple because it allows us to compute the surface genus by only counting the number of crossings and Seifert circles without advanced geometric or algebraic calculations. Seifert's algorithm has been a standard topological tool since 1934 because it simply requires a diagram and counting. For alternating knots, the algorithm is reliable because it always finds the true minimal genus, as confirmed by the trefoil and figure-eight cases. This reliability is significant because alternating knots are a large and well-studied family, so the algorithm provides a genus calculation shortcut for a substantial class of knots.

The investigation demonstrates that the algorithm is diagram sensitive, which is a weakness. Adding a Type 2 move to the trefoil increased $g_c(D)$ from 1 to 2, even though the knot is topologically unchanged. The algorithm cannot find the true genus when encountering redundant crossings. This sensitivity answers the first part of the research question by showing that the algorithm's output depends on diagram complexity rather than just the underlying knot. Another weakness is that the algorithm is locally blind because for the Whitehead double, it processes each clasp crossing as essential and increased the genus to 3 comparing to the true genus 1. This means the algorithm cannot see the simpler surface embedded inside the companion torus. This blindness causes the divergence $g_c(K) > g(K)$, answering the second part of the research question.

The prominent limitation of the essay is its narrow focus on satellite knots causing the inequality $g_c(K) > g(K)$, because Brittenham shows that non-alternating hyperbolic knots that are not satellites may also have $g_c(K) > g(K)$ (Brittenham), but it is too intricate to construct examples of such knots in this essay. Knots being satellites are not necessary for the genus divergence, so a complete classification of what causes knots to satisfy $g_c(K) > g(K)$ remains an open question. Another limitation is the manual counting method to find d and s. For knots with more than, say, fifteen crossings, counting Seifert circles by hand becomes impractical and prone

to error. An improvement in the methodology would be to write a computer program that automates Seifert's algorithm, which would eliminate clumsy counting errors and allow testing on large knots with more than ten crossings. In practice, topologists use algebraic invariants such as the Alexander and Jones polynomials, whose degrees bound the genus without requiring careful visual inspection.

Besides, we rely on Tripp's theorem for deriving the Whitehead double's canonical genus without verifying its proof. The proof uses advanced surgery techniques beyond the scope of this essay, but the theorem is well-established and frequently cited, so using it here is appropriate.

8. CONCLUSION

This essay asked: **To what extent does the output of Seifert's algorithm depend on diagram complexity, and under what structural conditions does the canonical genus diverge from the true knot genus?**

We achieve three conclusions. First, the algorithm's output $g_c(D)$ depends directly on the choice of diagram, as shown by adding a Type 2 move to the trefoil that increases the canonical genus from 1 to 2. Second, the alternating structure guarantees $g_c(K) = g(K)$, as shown by the hyperbolic figure-eight knot. Third, non-alternating structure can create a gap as shown by the Whitehead double of the trefoil with $g(K) = 1$ and $g_c(K) = 3$, even after minimizing over all diagrams. The satellite structure is one way to cause this divergence, but Brittenham proves that non-alternating hyperbolic knots can cause it too.

These findings are worthy because they show the fundamental limit of visual methods to differentiate knots, and in topology in general. Diagrams are vivid but can include complexity that is not inherently there. Seifert's algorithm works solely on 2D projections so it cannot distinguish topological complexity from diagrammatic complexity. This is why topologists use algebraic invariants, which make geometric information computable, alongside visual algorithms to distinguish knots effectively.

A thorough classification of knots with $g_c(K) > g(K)$ is still unformulated. This essay focuses on knots but not links, so extending the case studies to links would be a natural next step. Writing a program to automate Seifert's algorithm would allow testing genera across larger knots to be faster and would reveal how quickly the canonical genus grows with the crossing number.

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